

# MORE EXAMPLES OF MOTIVIC CELL STRUCTURES

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ABSTRACT. In this note, we describe motivic cell structures arising from the Białynicki-Birula decomposition. This provides a description of stable  $\mathbb{A}^1$ -homotopy types of smooth projective  $\mathbb{G}_m$ -varieties where the  $\mathbb{G}_m$ -action has isolated fixed points.

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## 1. INTRODUCTION

In this note, we describe motivic cell structures for some smooth projective schemes. The term motivic cell structure was coined in a paper of Dugger and Isaksen [DI05], as an  $\mathbb{A}^1$ -homotopy analogue of CW-structures in algebraic topology. Motivic cell structures describe how a scheme or a simplicial sheaf is built out of spheres (whenever that happens to be possible), and so motivic cell structures can in principle be used to compute arbitrary generalized cohomology theories. Several examples of motivic cell structures have been given in [DI05], and it is the goal of this note to add to the available list.

For a smooth compact manifold, one can use Morse functions and their associated gradient flows to obtain a CW-structure on the manifold. In the algebraic setting, there are no Morse functions available, any algebraic map  $X \rightarrow \mathbb{A}^1$  from a projective variety to the affine line is constant. However, one can interpret the gradient flow associated to a Morse function as an  $\mathbb{R}^\times$ -action on the manifold. In this formulation, the existence of CW-structures associated to Morse functions has an algebraic analogue - a cell structure exists for smooth projective varieties with a  $\mathbb{G}_m$ -action with isolated fixed points. For these varieties it is already known that the Białynicki-Birula filtration allows to produce splittings of motives of these varieties, cf. [Bro05]. We show here that the Białynicki-Birula filtration actually carries more information, which allows to extract a motivic cell structure. For this, however, we need to assume that the multiplicative group acts with isolated fixed points. For projective homogeneous varieties under split reductive groups, we even get unstable cell structures from the Schubert cell decomposition. This solves a question posed in [DI05, Remark 4.5].

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**Theorem 1.** *Let  $k$  be a field, and let  $X$  be a smooth projective variety equipped with an action of  $\mathbb{G}_m$  with isolated fixed points. Then  $X$  is stably cellular. If  $X$  is a projective homogeneous space under a split connected reductive group, then  $X$  is also unstably cellular.*

The motivic cell structures described above immediately induce splittings of the motives associated to these varieties. In the case of the smooth projective varieties of Theorem 1, the motivic splittings are the ones known from [Bro05]. But the motivic cell structures give complete descriptions of the stable homotopy types of the considered varieties, and could in principle be used to compute representable cohomology theories.

*Structure of the paper:* In Section 2, we recall the definition of motivic cell structures and collect a few preliminary results. In Section 3, we produce cell structures on smooth projective  $\mathbb{G}_m$ -varieties from the Białynicki-Birula decomposition. In Section 4, we deduce cellularity of reductive groups and their classifying spaces. Finally, in Section 5, we discuss consequences of the results.

## 2. PRELIMINARIES ON MOTIVIC CELL STRUCTURES

For the basics of  $\mathbb{A}^1$ -homotopy theory, we refer to [MV99]. In the sequel, the base scheme will be a field unless mentioned otherwise. The  $\mathbb{A}^1$ -homotopy theory is constructed as the  $\mathbb{A}^1$ -localization of the category of simplicial sheaves on the category  $\mathrm{Sm}_k$  of smooth schemes over the field  $k$  equipped with the Nisnevich topology.

Since we are interested in cell decompositions of varieties, we next recall the definition of motivic cell structures in the sense of Dugger and Isaksen, cf. [DI05, Definition 2.1].

**Definition 2.1.** *Let  $\mathcal{M}$  be a pointed model category and  $\mathcal{A}$  be a set of objects in  $\mathcal{M}$ . The class of  $\mathcal{A}$ -cellular objects is the smallest class of objects of  $\mathcal{M}$  such that*

- (C1) *every object of  $\mathcal{A}$  is  $\mathcal{A}$ -cellular,*
- (C2) *if  $X$  is weakly equivalent to an  $\mathcal{A}$ -cellular object, then  $X$  is  $\mathcal{A}$ -cellular,*
- (C3) *if  $D : I \rightarrow \mathcal{M}$  is a diagram such that each  $D_i$  is  $\mathcal{A}$ -cellular, then  $\mathrm{hocolim} D$  is  $\mathcal{A}$ -cellular.*

In the model category for  $\mathbb{A}^1$ -homotopy theory, we have spheres

$$S^{p,q} = S_s^{p-q} \wedge \mathbb{G}_m^{\wedge q},$$

where  $S_s^1$  is the simplicial circle. We will abbreviate  $\{S^{p,q} \mid p, q \in \mathbb{N}\}$ -cellular to cellular, as this is the only notion of cellularity we will use.

**2.1. Stable cell structures for varieties with cellular filtration.** We show how cellular filtrations on smooth varieties yield homotopy pushouts. The unusual twist in the proof is that one would usually use the closed subvarieties in the filtration to describe the cell structure. However, to avoid singularity-related problems, it is better to look at the complements of the subvarieties in the filtration. A similar trick has also been used by Zibrowius in his work on computation of Witt groups, cf. [Zib10]. Recall from [Bro05, Definition 3.1] that an *affine fibration* is a flat morphism  $\phi : X \rightarrow Z$  such that there exists a Zariski covering  $U_i$  of the base  $Z$  such that  $\phi|_{U_i} : U_i \times_Z X \cong U_i \times \mathbb{A}^n \rightarrow U_i$  is the projection away from the  $\mathbb{A}^n$ -factor.

**Proposition 2.2.** *Let  $X$  be a smooth scheme over an arbitrary base scheme  $S$ . Assume that there is a filtration of  $X$  by closed subschemes*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

Assume that for each  $i$  there is a smooth scheme  $Z_i$  and an affine fibration  $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$ . Then  $X$  is stably  $\mathcal{A}$ -cellular for

$$\mathcal{A} = \{S^{p,q} \wedge \mathrm{Th}(N_i) \mid p, q \in \mathbb{Z}, 0 \leq i < n\} \cup \{Z_n\},$$

where  $\mathrm{Th}(N_i)$  is defined via the homotopy cofibre sequence

$$X \setminus X_i \rightarrow X \setminus X_{i-1} \rightarrow \mathrm{Th}(N_i).$$

In particular, if  $S$  is smooth over an excellent Dedekind ring and all the  $X_i \setminus X_{i-1}$  are disjoint unions of copies of affine spaces, then  $X$  is stably cellular.

*Proof.* By assumption,  $(X \setminus X_{i-1}, X_i \setminus X_{i-1})$  is a smooth pair, since  $Z_i$  is assumed to be smooth, and  $X_i \setminus X_{i-1}$  is an affine fibration. By the homotopy purity theorem of Morel and Voevodsky, cf. [MV99, Theorem 3.2.23], we obtain a homotopy cofibre sequence

$$X \setminus X_i \rightarrow X \setminus X_{i-1} \rightarrow \mathrm{Th}(N_i).$$

Now we use [DI05, Lemma 2.5], which shows that if in a cofibre sequence  $A \rightarrow B \rightarrow C$  any two spaces are stably cellular, then so is the third. Applied to the above cofibre sequences, we inductively conclude that  $X$  is stably cellular. The base case is  $X \setminus X_{n-1} \simeq Z_n$ .

If  $X_i \setminus X_{i-1} \cong \bigsqcup_j \mathbb{A}^{n_j}$  is a finite disjoint union of copies of affine spaces, then because of the assumption, all normal bundles are trivial and we find

$$\mathrm{Th}(N_i) \simeq \bigvee_j S^{2n_j, n_j}.$$

This settles the final assertion.  $\square$

In particular, it is possible to see the attaching maps from a cellular filtration, at least stably: the suspension of the attaching map is  $\mathrm{Th}(N_i) \rightarrow \Sigma(X \setminus X_i)$ .

### 3. CELL STRUCTURES FROM TORUS ACTIONS

In this section, we use the Białynicki-Birula method to provide cell structures for some smooth projective varieties. This method has already been used to provide decompositions of motives, cf. [Bro05].

**3.1. The Białynicki-Birula decomposition.** The main geometric ingredient for the construction of cell decompositions of projective homogeneous varieties is the Białynicki-Birula decomposition. The following formulation can be found in [Bro05, Theorem 3.2]. There it is attributed to Białynicki-Birula, Iversen and Hesselink [BB73, BB76, Hes81, Ive72], for an explanation of the history, cf. [Bro05].

**Theorem 3.1.** *Let  $X$  be a smooth, projective variety over a field  $k$  equipped with an action of the multiplicative group  $\mathbb{G}_m$ . Then*

- (i) *The fixed point locus  $X^{\mathbb{G}_m}$  is a smooth, closed subscheme of  $X$ .*
- (ii) *There is a numbering  $X^{\mathbb{G}_m} = \coprod_{i=1}^n Z_i$  of the connected components of the fixed point locus, a filtration*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

*and affine fibrations  $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$ .*

- (iii) *The relative dimension  $a_i$  of the affine fibration  $\phi_i$  is the dimension of the positive eigenspace of the action of  $\mathbb{G}_m$  on the tangent space of  $X$  at an arbitrary point  $z \in Z_i$ . The dimension of  $Z_i$  is the dimension of  $TX_z^{\mathbb{G}_m}$ .*

**3.2. Stable cell structures for  $\mathbb{G}_m$ -varieties.** Now we use the Białynicki-Birula decomposition to obtain stable cell structures for  $\mathbb{G}_m$ -varieties with isolated fixed points.

**Proposition 3.2.** *Let  $X$  be a smooth, projective variety over a field  $k$ . If there exists an action of the multiplicative group  $\mathbb{G}_m$  on  $X$  which has isolated fixed points, then  $X$  is stably cellular.*

*Proof.* We use the Białynicki-Birula decomposition. In case the  $\mathbb{G}_m$ -action has only isolated fixed points, the fixed locus  $X^{\mathbb{G}_m}$  is a union of finitely many  $k$ -points  $Z_1, \dots, Z_n$ . Therefore, there is a filtration

$$X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$$

and the  $X_i \setminus X_{i-1}$  are isomorphic to affine spaces  $\mathbb{A}^{n_i}$ . The schemes  $X_i \setminus X_{i-1}$  are the sets of points  $x \in X$  such that  $\lim_{t \rightarrow 0} tx \in Z_i$ , where  $x \mapsto tx$  denotes the  $\mathbb{G}_m$ -action. It is then obvious that the open subvariety  $X \setminus X_i \hookrightarrow X \setminus X_{i-1}$  has complement isomorphic to  $\mathbb{A}^{n_i}$ . Now we apply Proposition 2.2 to this filtration.  $\square$

Now we will show that there is a great supply of varieties to which the above result applies.

**Definition 3.3.** *Let  $k$  be a field, and let  $G$  be a connected split reductive group. A normal  $G$ -variety  $X$  is called spherical, if some Borel subgroup of  $G$  has an open orbit on  $X$ .*

Particular examples of spherical varieties are projective homogeneous varieties, toric varieties, symmetric varieties and their wonderful compactifications.

**Proposition 3.4.** *Let  $G$  be a split reductive group with Borel subgroup  $B$ , and let  $X$  be a projective smooth spherical  $G$ -variety. Then there exists a group homomorphism  $\mathbb{G}_m \rightarrow B$  such that the induced  $\mathbb{G}_m$ -action on  $X$  has isolated fixed points.*

*Proof.* By definition, there are only finitely many  $B$ -orbits in  $X$ . Therefore, the induced action of the maximal torus  $T$  has finitely many fixed points. We show that there exists a morphism  $\mathbb{G}_m \rightarrow T$  which also has finitely many fixed points, which basically follows from Thomason's generic slice theorem [Tho86, Theorem 4.10]. Note that the base scheme  $S$  is the spectrum of a field,  $X$  is smooth projective over  $k$ , so the conditions are satisfied. The generic slice theorem then states that there exists an open affine subscheme  $U \subseteq X$  and a subtorus  $T' \hookrightarrow T$  such that  $U \cong T/T' \times U/T$ . Since  $T$  acts with isolated fixed points,  $T'$  is a strict subtorus – it is the generic stabilizer for  $U$ . In particular, any subgroup  $\mathbb{G}_m \subseteq T$  with  $\mathbb{G}_m \not\subseteq T'$  will act without fixed points on  $U$ . Now the complement  $X \setminus U$  is also a  $T$ -variety, and it has smaller dimension than  $X$ . Inductively, we find finitely many proper subtori  $T_i \subseteq T$  such that any subgroup  $\mathbb{G}_m \subseteq T$  avoiding these subtori  $T_i$  has the same fixed set as  $T$ . This concludes the proof.  $\square$

**Corollary 3.5.** *Let  $G$  be a connected split reductive group over  $k$ , and let  $X$  be a smooth projective spherical  $G$ -variety. Then  $X$  is stably cellular.*

*Proof.* By Proposition 3.4, there exists a  $\mathbb{G}_m$ -action with isolated fixed points on  $X$ . By Proposition 3.2,  $X$  is stably cellular.  $\square$

This in particular applies to wonderful compactifications of semisimple groups.

**3.3. Unstable cell structures for projective homogeneous spaces.** We have seen in the previous section that projective homogeneous varieties under split reductive groups are stably cellular. Here, we use slightly more precise information to obtain unstable cell structures. Recall that one way to produce the stable cell structures is the following result which provides a  $\mathbb{G}_m$ -action with isolated fixed points, cf. [KP00, Section 1].

**Theorem 3.6.** *Let  $G$  be a split semisimple group, let  $T$  be a maximal torus, and let  $P$  be a standard parabolic subgroup. Then the following assertions hold:*

- (i) *The  $T$ -action on  $G/P$  has isolated fixed points.*
- (ii) *Let  $\mathbb{G}_m \subseteq T$  correspond to an interior point of a Weyl-chamber. Then  $(G/P)^{\mathbb{G}_m} = (G/P)^T$  and the Białynicki-Birula decomposition obtained from the  $\mathbb{G}_m$ -action is an affine stratification of  $G/P$ .*

It is worth noting that the Białynicki-Birula stratification for the above  $\mathbb{G}_m$ -action coincides (up to the Weyl group action) with the Schubert cell stratification.

We are going to use the Schubert cell stratification to provide the unstable cells.

**Proposition 3.7.** *If  $X = G/P$  is homogeneous under a split connected reductive group  $G$ , then  $X$  is unstably cellular.*

*Proof.* As in Proposition 3.2, there is a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

and the  $X_i \setminus X_{i-1}$  are isomorphic to affine spaces  $\mathbb{A}^{n_i}$ . Via the identification with the Schubert cell stratification, we can describe the cells as cosets  $BwP/P$  with  $w$  running through the Weyl group  $W$  of  $G$ .

The inclusion  $X \setminus X_i \hookrightarrow X \setminus X_{i-1}$  has complement isomorphic to  $\mathbb{A}^{n_i}$  and we fix an isomorphism mapping  $0 \in \mathbb{A}^{n_i}$  to the  $\mathbb{G}_m$ -fixed point  $x_i$  of  $X_i$ .

There is a unique  $\mathbb{G}_m$ -fixed point  $z \in X$  of maximal index, this is the fixed point contained in  $X \setminus X_n = Bw_0P/P$  with  $w_0$  the longest element of the Weyl group.

Now there exists  $w \in W$  such that  $wx_i = z$ . This maps the  $\mathbb{A}^{n_i}$  isomorphically to an affine subspace of the big cell  $X \setminus X_n$ . It follows that

$$(w^{-1}(X \setminus X_n)) \cap X \setminus X_i \cong \mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i}$$

Then the following is a Zariski covering of  $X \setminus X_{i-1}$ :

$$\begin{array}{ccc} \mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i} & \longrightarrow & X \setminus X_i \\ \downarrow & & \downarrow \\ \mathbb{A}^{\dim X} & \longrightarrow & X \setminus X_{i-1}. \end{array}$$

But this implies that we have a homotopy cofibre sequence

$$\mathbb{A}^{\dim X - n_i} \setminus \{0\} \rightarrow X \setminus X_i \rightarrow X \setminus X_{i-1}$$

These cofibre sequences provide the unstable cell decomposition of  $X$ .  $\square$

Note that this result holds over general base schemes as soon as the split reductive group  $G$  is defined over  $\mathbb{Z}$ .

**3.4. Example: Even-dimensional Quadrics.** As an example how the above works, we discuss the case of the  $2n$ -dimensional projective quadric  $Q_{2n} \subseteq \mathbb{P}^{2n+1}$  defined by the vanishing of the split symmetric bilinear form

$$V \left( \sum_{i=0}^n x_i y_i \right) = Q_{2n} \subseteq \mathbb{P}^{2n+1}.$$

This is a homogeneous space under the split group  $PSO(2n+2)$ . There are two important closed subvarieties,

$$Z_x = \{x_0 = x_1 = \cdots = x_n = 0\}, \quad Z_y = \{y_0 = y_1 = \cdots = y_n = 0\},$$

which arise from the Białynicki-Birula decomposition associated to the  $\mathbb{G}_m$ -action

$$\mathbb{G}_m \times Q_{2n} \rightarrow Q_{2n} : (t, [x_0 : \cdots : x_n : y_0 : \cdots : y_n]) \mapsto [tx_0 : \cdots : tx_n : y_0 : \cdots : y_n].$$

We have  $Z_x \cong Z_y \cong \mathbb{P}^n$ . The obvious projection  $\pi : Q_{2n} \setminus Z_x \rightarrow Z_y$  onto the  $x$ -coordinates is a rank  $n$  vector bundle. Therefore,  $Q_{2n} \setminus Z_x \simeq \mathbb{P}^n$  for which we have the usual cell structure.

Now  $Z_x \cong \mathbb{P}^n$  has a filtration by projective spaces  $\mathbb{P}^k$ ,  $0 \leq k \leq n$ , which are given by the equations

$$x_{k+1} = x_{k+2} = \cdots = x_n = 0.$$

We denote by  $V_k$  the subvariety of  $Q_{2n}$  given by  $x_k \neq 0$ , which is isomorphic to  $\mathbb{A}^{2n}$ . The intersection  $\mathbb{P}^k \cap V_k$  is then isomorphic to affine  $k$ -dimensional subspace  $\mathbb{A}^k$ . Thus we have a Zariski covering

$$\begin{array}{ccc} \mathbb{A}^{2n-k} \setminus \{0\} \simeq \mathbb{A}^{2n} \setminus \mathbb{A}^k & \longrightarrow & Q_{2n} \setminus \mathbb{P}^k \\ \downarrow & & \downarrow \\ \mathbb{A}^{2n} \cong V_k & \longrightarrow & Q_{2n} \setminus \mathbb{P}^{k-1}. \end{array}$$

These yield homotopy cofibre sequences

$$\mathbb{A}^{2n-k} \setminus \{0\} \rightarrow Q_{2n} \setminus \mathbb{P}^k \rightarrow Q_{2n} \setminus \mathbb{P}^{k-1}$$

which explain how to successively attach cells to  $Q_{2n} \setminus Z_x$  to finally obtain  $Q_{2n}$ . This recovers the classical cell decomposition known for the projective even-dimensional quadrics.

**3.5. Remarks on the general case.** Finally, we want to discuss the case of arbitrary smooth projective  $\mathbb{G}_m$ -varieties. As formulated in Theorem 3.1, the fixed point locus  $X^{\mathbb{G}_m}$  is a smooth closed subscheme, and the successive strata  $X_i \setminus X_{i-1}$  are vector bundle torsors over components of  $X^{\mathbb{G}_m}$ . We can then apply Proposition 2.2 to see that only Thom spaces over components of the fixed point locus  $X^{\mathbb{G}_m}$  are needed in order to reconstruct the stable homotopy type of  $X$ .

In particular, we get the following refinement of the motivic decompositions for isotropic projective homogeneous varieties given by Chernousov, Gille and Merkurjev [CGM05] resp. by Brosnan [Bro05].

**Proposition 3.8.** *Let  $k$  be a field, and let  $X$  be an isotropic projective homogeneous variety under a reductive group  $G$ . Then the Białynicki-Birula filtration*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

*induces homotopy cofibre sequences*

$$X \setminus X_i \rightarrow X \setminus X_{i-1} \rightarrow \mathrm{Th}(Z)$$

*where  $Z$  is a quasi-homogeneous projective variety under the anisotropic kernel of  $G$ .*

*Proof.* This follows from the geometric analysis of the Białynicki-Birula filtration in [Bro05, Section 4] and the argument in Proposition 3.2.  $\square$

This homotopy colimit description of isotropic projective homogeneous varieties in the stable homotopy category refines the motivic decomposition known from [CGM05] and [Bro05]. We see that to describe the stable homotopy types of isotropic projective homogeneous spaces, we need stable homotopy types of Thom

spaces over anisotropic projective homogeneous varieties. It is not clear to me if the normal bundles of anisotropic varieties are trivial, i.e. if there are  $\mathbb{A}^1$ -local weak equivalences  $\mathrm{Th}(Z) \simeq S^{2n,n} \wedge Z_+$ .

#### 4. CELLULARITY FOR GROUPS AND CLASSIFYING SPACES

The cell structure for projective homogeneous varieties can be lifted to show cellularity for split reductive groups. The following was also proved in [Mor12b].

**Proposition 4.1.** *Let  $G$  be a connected split reductive group. Then  $G$  is stably cellular.*

*Proof.* Let  $B$  be a Borel subgroup of  $G$ . Then there is a smooth morphism  $\pi : G \rightarrow G/B$ . Now let

$$G/B = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

be the filtration yielding the cell structure of  $G/B$  from Proposition 3.7. We obtain a filtration

$$G = Y_n \supset Y_{n-1} \supset \cdots \supset Y_0 \supset Y_{-1} = \emptyset$$

by setting  $Y_i = \pi^{-1}(X_i)$ . As in the proof of Proposition 3.7, we obtain a Zariski covering

$$\begin{array}{ccc} \pi^{-1}(\mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i}) \cong \mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i} \times B & \longrightarrow & Y \setminus Y_i \\ \downarrow & & \downarrow \\ \pi^{-1}(\mathbb{A}^{\dim X}) \cong \mathbb{A}^{\dim X} \times B & \longrightarrow & Y \setminus Y_{i-1}. \end{array}$$

The spaces  $\pi^{-1}(\mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i})$  and  $\pi^{-1}(\mathbb{A}^{\dim X})$  are  $B$ -bundles over  $\mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i}$  and  $\mathbb{A}^{\dim X}$  respectively. The  $B$ -bundle  $\pi^{-1}(\mathbb{A}^{\dim X})$  can be factored as composition of a  $T$ -bundle followed by a vector bundle. Therefore,  $\pi^{-1}(\mathbb{A}^{\dim X}) \rightarrow \mathbb{A}^{\dim X}$  is a trivial  $B$ -bundle. The bundle  $\pi^{-1}(\mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i}) \rightarrow \mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i}$  is the restriction of the former bundle along the inclusion  $\mathbb{A}^{\dim X} \setminus \mathbb{A}^{n_i} \hookrightarrow \mathbb{A}^{\dim X}$ , hence also trivial. This justifies the isomorphisms in the above diagram.

Stable cellularity now follows by induction, and the fact that stable cellularity is preserved under products, cf. [DI05, Lemma 3.4].  $\square$

This approach can be used to reproduce the motivic decompositions of split reductive groups obtained by Biglari [Big05].

From the cellularity of the group  $G$  we can deduce the cellularity of the classifying space  $BG$ . Here we consider the classifying space of Nisnevich locally trivial torsors.

**Proposition 4.2.** *Let  $G$  be a connected split reductive group over an infinite field  $k$ . Then  $BG$  is stably cellular.*

*Proof.* We use the proof of [DF96, Theorem 2.D.11]. We start with the fibre sequence  $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G \rightarrow E \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$ . This is an  $\mathbb{A}^1$ -local fibre sequence, cf. [Wen11a, Theorem 4.7].

We define a sequence of fibrations  $F_i \rightarrow E_i \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  by setting

$$E_0 = E \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G, F_0 = \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G, E_{i+1} = E_i \cup C F_i,$$

and  $F_{i+1}$  is the homotopy fibre of the obvious morphism  $E_{i+1} \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  in the  $\mathbb{A}^1$ -local category. By Ganea's theorem [Wen11b, Proposition 2.22], we have in the simplicial model structure

$$F_{i+1} \simeq F_i * \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G \simeq \Sigma(F_i \wedge \mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G).$$

Inductively, we conclude that the simplicial connectivity of  $F_{i+1}$  is at least  $i$ . By Morel's  $\mathbb{A}^1$ -connectivity theorem [Mor12a, Theorem 15], the space  $F_{i+1}$  is  $i$ - $\mathbb{A}^1$ -connected. Since  $B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  is  $\mathbb{A}^1$ -local, the homotopy fibre of  $E_{i+1} \rightarrow B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  is  $\mathbb{A}^1$ -weakly equivalent to  $F_{i+1}$ , this is a consequence of [Wen10, Theorem 1].

From the assumption and Proposition 4.1, it follows that  $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  is stably cellular, and inductively, we conclude that  $E_i$  and  $F_i$  are stably cellular. This follows since  $E_{i+1}$  is constructed as homotopy cofibre of  $F_i \rightarrow E_i$ , and as seen above,  $F_{i+1}$  is the suspension of a smash-product of  $F_i$  and  $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$ .

We denote by  $F_{\infty} \rightarrow E_{\infty} \rightarrow B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  the colimit of the fibre sequences above. This is a simplicial fibre sequence, by homotopy distributivity, cf. [Wen11b, Proposition 2.17]. Since  $B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  is  $\mathbb{A}^1$ -local, the homotopy fibre of  $E_{\infty} \rightarrow B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$  is  $\mathbb{A}^1$ -weakly equivalent to  $F_{\infty}$ . But as noted above,  $F_{\infty}$  is simplicially contractible. Therefore,  $B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G \simeq \mathrm{hocolim} E_i$ , which implies stable cellularity of  $B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1} G$ .  $\square$

**Remark 4.3.** *Alternatively, the construction of the classifying space shows directly that  $BG$  is  $G$ -cellular in the simplicial model structure. This implies that  $BG$  is also  $G$ -cellular in the  $\mathbb{A}^1$ -local model structure. But Proposition 4.1 implies that  $\mathbb{A}^1$ -locally,  $G$  is cellular, so  $BG$  is cellular.*

## 5. CONSEQUENCES

**5.1. Motivic Decompositions.** As a consequence of the motivic cell structures developed, we obtain stable homotopy theory proofs of the motivic decompositions in [Bro05]. In this section, motives are objects in Voevodsky's category  $DM(k)$ . The functor  $\mathrm{Sm}_k \rightarrow DM_k$  extends to a functor  $\mathcal{SH}(k) \rightarrow DM(k)$ , so in fact we can consider motives associated to arbitrary stable homotopy types.

**Corollary 5.1.** *Let  $X$  be a smooth projective variety over a field  $k$ . Assume there is a  $\mathbb{G}_m$ -action on  $X$  with isolated fixed points. Then the motive of  $X$  splits as a direct sum of Tate motives:*

$$M(X) \cong \bigoplus_i \mathbb{Z}(n_i)[2n_i].$$

*Proof.* From Proposition 3.2, we obtained homotopy cofibre sequences

$$X \setminus X_i \rightarrow X \setminus X_{i-1} \rightarrow S^{2(\dim X - n_i), \dim X - n_i}$$

describing the stable  $\mathbb{A}^1$ -homotopy type of  $X$ . These induce distinguished triangles in  $\mathcal{SH}(k)$ , and therefore also distinguished triangles in  $DM(k)$ . Now we prove the result inductively for the motives  $M(X \setminus X_i)$ . For convenience, we rewrite the above distinguished triangles as

$$M(S^{2(\dim X - n_i) - 1, \dim X - n_i}) \rightarrow M(X \setminus X_i) \rightarrow M(X \setminus X_{i-1}).$$

By inductive assumption  $M(X \setminus X_i)$  is a direct sum of Tate motives of the form  $\mathbb{Z}(n_j)[2n_j]$ . It then suffices to show that the morphism

$$M(S^{2(\dim X - n_i) - 1, \dim X - n_i}) \rightarrow M(X \setminus X_i)$$

is trivial, hence induces a splitting

$$M(X \setminus X_{i-1}) \cong M(S^{2(\dim X - n_i), \dim X - n_i}) \oplus M(X \setminus X_i).$$

Recall that in the Białyński-Birula filtration, the dimension of the big cell in the step  $X_i$  equals the dimension of  $X_i$ , so for  $i > j$ , we have  $n_i > n_j$ . So all the



Tate motives in  $M(X \setminus X_i)$  have weight at most  $\dim X - n_i$ . So the composition of the morphism

$$M(S^{2(\dim X - n_i) - 1, \dim X - n_i}) \rightarrow M(X \setminus X_i)$$

with a projection onto a summand of  $M(X \setminus X_i)$  is of the form  $\mathbb{Z}(m)[2m - 1] \rightarrow \mathbb{Z}(n)[2n]$  with  $m \geq n$ . This is always 0.  $\square$

For projective homogeneous varieties, one can obtain the various weights of the summands from the Hasse diagram, cf. [Sem06, Bro05].

**5.2. Morel's Decomposition Theorem.** We can also obtain decomposition theorems for projective homogeneous varieties in the  $\mathbb{A}^1$ -stable homotopy theory. For this, we recall the following splitting of the  $\mathbb{A}^1$ -sphere spectrum, cf. [Mor06a]: Denoting by  $\epsilon \in [S^0, S^0]$  the automorphism induced by the permutation of factors in  $\mathbb{G}_m \wedge \mathbb{G}_m$ , there are orthogonal idempotents

$$e_- = \frac{\epsilon + 1}{2}, \quad e_+ = \frac{\epsilon - 1}{2}$$

acting on  $S^0[\frac{1}{2}]$ . These orthogonal idempotents yield a splitting

$$SH(S)_{\mathbb{Q}} \cong SH(S)_{\mathbb{Q}_+} \times SH(S)_{\mathbb{Q}_-}.$$

The motivic cell structures developed in this note allow to produce descriptions of  $\mathbb{Q}_+$  and  $\mathbb{Q}_-$ -local  $\mathbb{A}^1$ -stable homotopy types of projective homogeneous varieties.

The same proof as for the motivic decompositions imply  $\mathbb{Q}_+$ -decompositions provided that there are no morphisms  $S^{2n-1, n} \rightarrow S^{2k, k}$  for  $k < n$ .

**Proposition 5.2.** *Let  $X$  be a smooth projective variety over a field  $k$ . Assume there is a  $\mathbb{G}_m$ -action on  $X$  with isolated fixed points. Assume furthermore that we have*

$$\mathrm{Hom}_{SH(k) \otimes \mathbb{Q}_+}(S^{2i-1, i}, S^{2j, j}) = 0$$

for all  $i > j + 1$ . Then the  $\mathbb{Q}_+$ -localization of  $X$  splits as a wedge of  $(2n, n)$ -spheres

$$X \simeq_{\mathbb{Q}_+} \bigvee_i S^{2n_i, n_i}.$$

*Proof.* By [Mor12a, Corollary 3.43], we have

$$\mathrm{Hom}_{SH(k)}(S^{2i-1, i}, S^{2j, j}) = 0$$

for all  $i \leq j$  and

$$\mathrm{Hom}_{SH(k)}(S^{2i-1, i}, S^{2j-2, j-1}) = K_{-1}^{MW}(k).$$

In the latter group, every element is a multiple of the Hopf map, i.e. the morphism  $K_0^{MW}(k) \rightarrow K_{-1}^{MW}(k)$  induced by multiplication by  $\eta$  is surjective. But since  $\epsilon\eta = \eta$  in  $K^{MW}$ , we find  $e_+\eta = 0$ , so the Hopf map is annihilated in the  $\mathbb{Q}_+$ -localization. Therefore all morphisms  $S^{2i-1, i} \rightarrow S^{2i-2, i-1}$  are  $\mathbb{Q}_+$ -null. By the assumption above, we then have

$$\mathrm{Hom}_{SH(k) \otimes \mathbb{Q}_+}(S^{2i-1, i}, S^{2j, j}) = 0$$

for all  $i$  and  $j$ .

Now the same proof as for Corollary 5.1 yields the claim.  $\square$

In particular, under the assumptions of the proposition, we can decompose a projective homogeneous variety  $G/P$  in the  $\mathbb{Q}_+$ -localization into a wedge of  $S^{2n, n}$ -spheres indexed by the vertices of the Hasse diagram.

For the  $\mathbb{Q}_-$ -localization, we get the stable homotopy types of the real realization. This actually works in  $SH(k) \otimes \mathbb{Z}[\frac{1}{2}]_-$ .

**Proposition 5.3.** *Let  $k$  be a field, and let  $X$  be a homotopy colimit of  $S^{p,q}$ -spheres. Then  $X$  is already  $\{S^{n,0} \mid n \in \mathbb{Z}\}$ -cellular in the  $\mathbb{Z}[\frac{1}{2}]_-$ -local stable homotopy category. In particular, if  $k$  has a real embedding  $k \hookrightarrow \mathbb{R}$ , there is a  $\mathbb{Z}[\frac{1}{2}]_-$ -local weak equivalence  $X \simeq X(\mathbb{R})$ , where  $X(\mathbb{R})$  is the (locally constant) simplicial sheaf on  $\mathrm{Sm}_k$  associated to the simplicial set  $X(\mathbb{R})$ .*

*Proof.* The assertion follows immediately from the fact that  $S^0 \simeq \mathbb{G}_m$  in  $\mathcal{SH}(k) \otimes \mathbb{Z}[\frac{1}{2}]_-$ , cf. [Mor06a, Remark 1.6].  $\square$

The  $\mathbb{Z}[\frac{1}{2}]_-$ -local stable homotopy category can then serve as a replacement of real realization of cellular varieties over fields which do not have real embeddings. This could be used to prove indecomposability of stable homotopy types even in situations where no real or complex realization is available.

**5.3.  $\mathbb{A}^1$ -homology and rigidity.** We shortly recall from [Mor12a] the definition of  $\mathbb{A}^1$ -homology: to a simplicial sheaf  $X$  one can associate corresponding sheaf of chain complexes. The sheaves of homology groups of the corresponding fibrant replacement in the category of Nisnevich sheaves of abelian groups on  $\mathrm{Sm}_k$  are called  $\mathbb{A}^1$ -homology sheaves and denoted by  $\mathbf{H}_\bullet^{\mathbb{A}^1}(X, \mathbb{Z})$ . From the  $\mathbb{A}^1$ -derived category of Nisnevich sheaves of abelian groups on  $\mathrm{Sm}_k$  one can pass to the  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -derived category by formally inverting the “Tate object”  $C_*^{\mathbb{A}^1}(\mathbb{P}^1)[-2]$ . For more details on the definition cf. [AH11]. The cohomology in the  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -derived category is denoted by  $\mathbf{H}_\bullet^{s\mathbb{A}^1}(X, \mathbb{Z})$ . The latter are sheaves called the  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homology sheaves.

Note that the homotopy pushouts produced from the cellular filtrations induce corresponding long exact sequences in  $\mathbb{A}^1$ -homology. The motivic cell structures therefore can be used to compute  $\mathbb{A}^1$ -homology of smooth projective spherical varieties in terms of  $\mathbb{A}^1$ -homology of the spheres  $S^{2n,n}$ . The  $\mathbb{G}_m$ -stable  $\mathbb{A}^1$ -homology has the advantage that it has suspension isomorphisms not just for suspension with  $S^{n,0}$ , but also for suspension with  $S^{2n,n}$ .

The  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homology of a variety with motivic cell structure can then be computed from the cell structure: the homotopy cofibre sequences  $X \setminus X_i \rightarrow X \setminus X_{i-1} \rightarrow \mathrm{Th}(N_i)$  from Proposition 2.2 induce long exact sequences of  $\mathbb{A}^1$ -homology

$$\cdots \rightarrow \mathbf{H}_\bullet^{\mathbb{A}^1}(X \setminus X_i, \mathbb{Z}) \rightarrow \mathbf{H}_\bullet^{\mathbb{A}^1}(X \setminus X_{i-1}, \mathbb{Z}) \rightarrow \mathbf{H}_\bullet^{\mathbb{A}^1}(\mathrm{Th}(N_i), \mathbb{Z}) \rightarrow \cdots$$

If the complements in the cellular filtration are unions of affine spaces, then  $\mathrm{Th}(N_i) \simeq \bigvee_j S^{2n_j, n_j}$ , and we get a long exact sequence (in  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homology):

$$\cdots \rightarrow \mathbf{H}_\bullet^{s\mathbb{A}^1}(X \setminus X_i, \mathbb{Z}) \rightarrow \mathbf{H}_\bullet^{s\mathbb{A}^1}(X \setminus X_{i-1}, \mathbb{Z}) \rightarrow \bigoplus_j \mathbf{H}_{\bullet+n_j}^{s\mathbb{A}^1}(\mathrm{Spec}(k), \mathbb{Z}) \rightarrow \cdots$$

Informally, one would like to state the result as a module isomorphism

$$\mathbf{H}_\bullet^{s\mathbb{A}^1}(X, \mathbb{Z}) \cong H_\bullet(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbf{H}_\bullet^{s\mathbb{A}^1}(\mathrm{Spec} k, \mathbb{Z}).$$

This is, however, not accurate, since the attaching maps in  $\mathbb{A}^1$ -homology have much more information, and the passage to the complex realization loses a lot of this information. Nevertheless, for varieties with a motivic cell structure, one can rather easily evaluate the  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homology, provided one uses the  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homology of the base field as a black box. Note that similar arguments hold for reductive groups and their classifying spaces by Section 4.

We finally want to note the following:

**Corollary 5.4.** *Let  $X$  be a smooth projective variety over  $k$  with a  $\mathbb{G}_m$ -action with isolated fixed points. Then the  $\mathbb{A}^1$ -chain complex of  $X$  is of mixed Tate type in the sense of [Mor12b, Definition 4.7].*

In light of [Mor12b, Theorem 12], it is reasonable to conjecture that rigidity also holds for  $\mathbb{A}^1$ -homotopy of the varieties in the above corollary, even though they are not  $\mathbb{A}^1$ -simply-connected. This would imply that the  $\mathbb{A}^1$ -homotopy and  $\mathbb{A}^1$ -homology of these varieties over algebraically closed fields agrees with the homotopy and homology of the complex realizations, at least with finite coefficients away from the characteristic.

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